SUMMARY OF CARTOGRAPHIC PROJECTIONS

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Projections. Equivalent and conformal projections

A projection is an application from the points (ϕ, λ) of the sphere or ellipsoid into the points (E, N) of the plane. Let the formulas be

$$\mathrm{E} = \mathrm{E}(\phi, \lambda)$$
$$\mathrm{N} = \mathrm{N}(\phi, \lambda)$$

We will soon make use of the Jacobian matrix:

$$\mathrm{J} = \begin{pmatrix} \dfrac{1}{\cos\phi}\dfrac{\partial \mathrm{E}}{\partial \lambda} & \dfrac{\partial \mathrm{E}}{\partial \phi} \\ \dfrac{1}{\cos\phi}\dfrac{\partial \mathrm{N}}{\partial \lambda} & \dfrac{\partial \mathrm{N}}{\partial \phi} \end{pmatrix}$$

It is well known that any projection will produce some deformations. Two major choices are made with respect to it:

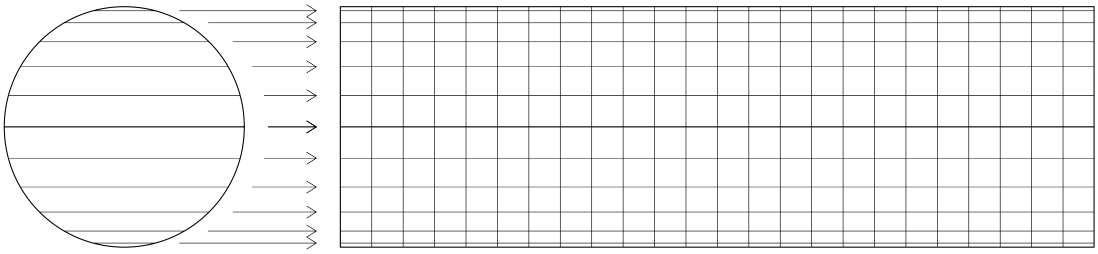
1. *Area preserving projections.* These projections are called *equivalent* and satisfy the relation
$$|\mathrm{J}| = \frac{1}{\cos\phi}\left(\frac{\partial \mathrm{E}}{\partial \lambda}\frac{\partial \mathrm{N}}{\partial \phi} - \frac{\partial \mathrm{E}}{\partial \phi}\frac{\partial \mathrm{N}}{\partial \lambda}\right) = \mathrm{R}^2$$

2. *Angle preserving projections.* These are called *conformal* and satisfy the relation
$$\frac{1}{\cos\phi}\frac{\partial \mathrm{E}}{\partial \lambda} = \frac{\partial \mathrm{N}}{\partial \phi},$$
$$\frac{\partial \mathrm{E}}{\partial \phi} = -\frac{1}{\cos\phi}\frac{\partial \mathrm{N}}{\partial \lambda}.$$

It is to be understood that these relations are satisfied for every pair (ϕ, λ). The value R is the Earth radius. If working with an ellipsoid the expressions become a little more complicated, but the principle is the same.

Example: The **Lambert equivalent** projection, pictured in the next page, can be expressed by the formulas:

$$\mathrm{E} = \mathrm{R}\lambda,$$
$$\mathrm{N} = \mathrm{R}\sin\phi.$$

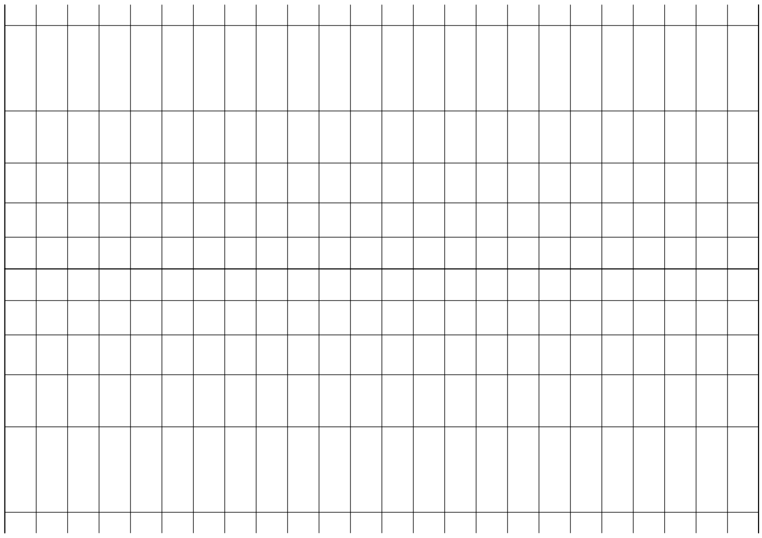


We compute J:

$$\begin{pmatrix} \dfrac{\mathrm{R}}{\cos\phi} & 0 \\ 0 & \cos\phi \mathrm{R} \end{pmatrix}$$

It satisfies $|\mathrm{J}| = \mathrm{R}^2$ and so it is area preserving.

The **Mercator** projection:



is expressed by

$$\begin{aligned} \mathrm{E} &= \mathrm{R}\lambda, \\ \mathrm{N} &= \mathrm{R}\log\tan\left(\tfrac{\pi}{4} + \tfrac{\phi}{2}\right). \end{aligned}$$

The same parallels than in the Lambert projection are shown for comparison.

In order to compute $\partial\mathrm{N}/\partial\phi$ we first perform some trigonometric manipulations:

$$\begin{aligned} \tan\left(\tfrac{\pi}{4} + \tfrac{\phi}{2}\right) &= \frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}} = \frac{(1+\tan\frac{\phi}{2})^2}{1-\tan^2\frac{\phi}{2}} = \frac{1+\frac{1-\cos\phi}{1+\cos\phi}+2\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}}{1-\frac{1-\cos\phi}{1+\cos\phi}} \\ &= \frac{1+\cos\phi+1-\cos\phi+2\sqrt{(1+\cos\phi)(1-\cos\phi)}}{1+\cos\phi-(1-\cos\phi)} \\ &= \frac{2\left(1+\sqrt{1-\cos^2\phi}\right)}{2\cos\phi} = \frac{1+\sin\phi}{\cos\phi} \end{aligned}$$

So

$$\frac{\partial N}{\partial \phi} = \frac{d}{d\phi}\left(R \log \frac{1+\sin\phi}{\cos\phi}\right) = R\left(\frac{\cos\phi}{1+\sin\phi} - \frac{-\sin\phi}{\cos\phi}\right)$$
$$= R\,\frac{\cos^2\phi + \sin^2\phi + \sin\phi}{(1+\sin\phi)\cos\phi} = \frac{R}{\cos\phi}.$$

Therefore,

$$J = \begin{pmatrix} \dfrac{R}{\cos\phi} & 0 \\ 0 & \dfrac{R}{\cos\phi} \end{pmatrix}$$

and the conditions for conformance are satisfied.

Actually, the way N is defined in the Mercator projection is “that the projection be conformal”, which leads immediately to

$$\frac{dN}{d\phi} = \frac{R}{\cos\phi},$$

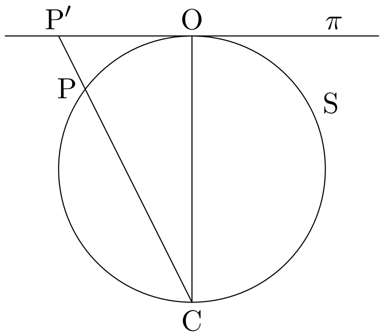
and it is therefrom that the explicit formula for N is found by integration.

Some projections

We have already seen the Lambert equivalent and the Mercator projections. We now see some others.

Stereographic

This is, together with the Mercator one, the single most important conformal projection. It is obtained by placing the projection plane tangent to the sphere at a point of it and projecting form the antipodal point the points of the sphere onto the plane. The plane has to be seen “from above”. Otherwise the map becomes reversed.

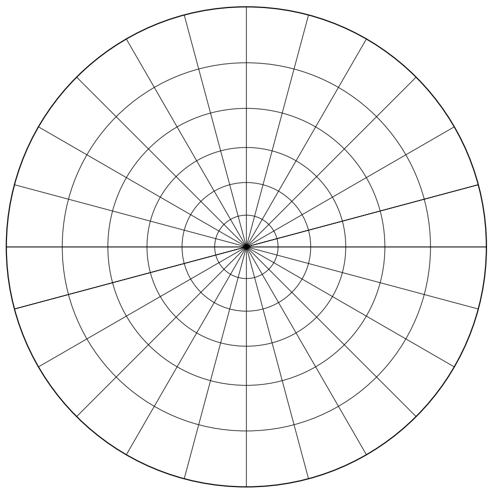


In the previous figure, the point P is mapped to the point P′.

When thought as a tridimensional transformation, it is an inversion centered at C that transforms the sphere S into the plane π. We can immediately apply all the properties of inversions. The most important:

- *The projection is conformal*
- *Circles* (including great circles) *are mapped into circles or straight lines.* The ones that are mapped into straight lines are those passing trough the inversion center; any other is mapped to a circle.

If we choose the tangent point to be one of the poles, meridians are mapped to straight lines. The following picture displays the stereographic projection of one hemisphere.



The distance in the projection from a point P' to the central point is

$$2\mathrm{R}\tan\frac{\theta}{2},$$

where θ is the angle from the tangent point to the point P on the sphere. If the tangent point is the North pole, then $\theta = \pi/2 - \phi$.

Lambert conformal conic

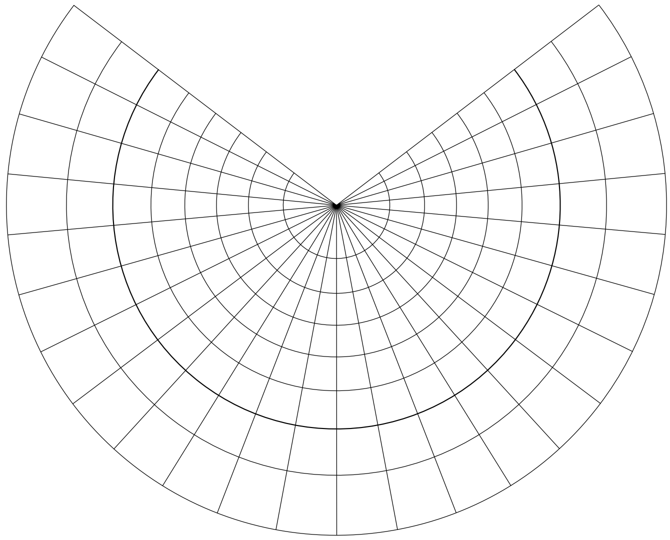
In this projection parallels are mapped into concentric arcs of circle of a given angle α. The value of α together with the requirement of conformality completely defines the projection (up to size).

It can be shown that there is a parallel where the scale of the map is minimum. Let it be ϕ_0. It can be shown that

$$\phi_0 = \arcsin\frac{\alpha}{2\pi}.$$

Parallels increase their separation as we move further from the central parallel ϕ_0, in the same way that they do with respect to the equator in the Mercator projection.

The following image pictures the Lambert projection corresponding to $\phi_0 = 45°$, from the North pole to latitude 30° South.



In order to give the formulas for the E and N coordinates, let the *isometric latitude*, Φ, be defined as the North coordinate corresponding to ϕ in the Mercator projection, divided by the Earth radius:

$$\Phi = \log \tan(\tfrac{\pi}{4} + \tfrac{\phi}{2}).$$

The formulas for the Lambert projection of central parallel ϕ_0 are

$$r_0 = \mathrm{R} \operatorname{cotan} \phi_0, \qquad r = r_0 e^{-\sin\phi_0(\Phi-\Phi_0)};$$

$$\begin{aligned} \mathrm{E} &= r \sin(\lambda \sin \phi_0), \\ \mathrm{N} &= r_0 - r\cos(\lambda \sin \phi_0). \end{aligned}$$

If we let $\phi_0 = 90°$ we get the stereographic projection, and for $\phi_0 = 0$ it is the Mercator projection.

The scale factor

As a result of the projection, a set of (E, N) coordinates is obtained. If we further divide these values by some constant we get a map at the corresponding scale. But we know that maps deform different regions of the Earth in different ways. Since equivalent projections preserve areas it makes sense to say that the map obtained by dividing each coordinate by 100 000, for instance, has a scale of 1 : 100 000. It should nevertheless be noted that, even if any figure will be transformed by the projection into a figure of equal area, it may be stretched in one direction and shrunk in the other.

For projections which are not equivalent the scale will vary from one point to another. This effect is very well known for the Mercator projection, were for instance

Greenland appears greater than South America. Within a map in an equivalent projection at a certain scale E, areas are scaled by a factor E^2. The scaling on areas is well defined and can be used to define the scale at every point for any projection.

Let S be the area of a figure Ω on the Earth surface and S' its area on the map. Call δ the diameter of Ω. The diameter of a general figure is defined as the maximum distance between any two points of the figure. The scale at a point P is defined by

$$\mathrm{E}^2 = \lim_{\delta \to 0} \frac{\mathrm{S}'}{\mathrm{S}}, \quad \mathrm{P} \in \Omega.$$

That is, we take ever smaller figures around P and the quotient S'/S will tend to a limit, which is E^2.

For conformal projections, a small segment arising from P is scaled the same amount in any direction, so for these projections the scale E can be obtained by

$$\mathrm{E} = \frac{\mathrm{d}s'}{\mathrm{d}s},$$

where s and s' are the lengths of corresponding segments on the Earth surface and on the map, arising from P and its projection respectively.

For the conformal projections considered so far, if the coordinates are multiplied by the scale E, this value will equal the actual scale along its distinguished element, i.e., the Equator for the Mercator projection, the Pole of the Stereographic projection and the parallel ϕ_0 in the Lambert conic projections. And in all the cases the scale increases as we move away form those elements.

Again in all the cases, the value of the scale is constant within parallels and is easily computed along the direction of λ. It equals the scale at the distinguished element times a factor k:

$$k_{\mathrm{Mercator}} = \frac{1}{\cos\phi}, \qquad k_{\mathrm{Stereog.}} = \frac{2}{1+\sin\phi},$$

$$k_{\mathrm{Lambert}} = \frac{\cos\phi_0}{\cos\phi}\, e^{-\sin\phi_0(\Phi-\Phi_0)}.$$

If the nominal scale of a map is that of the distinguished element, the map will elsewhere have a greater scale. In order to avoid that, and to provide a nominal value which is a mean value of the map scale, this nominal value is made to coincide with the scale at intermediate parallels.

Example: Compute the value which all the coordinates arising from the given formulas of the Mercator projection need to be multiplied by (besides the scale itself) in order that the nominal scale equal the actual scale for $\phi = 20°$.

If the scale at the Equator is E, at a latitude of 20° is

$$k(20°)\mathrm{E} = \frac{1}{\cos 20°}\mathrm{E} = 1.0642\mathrm{E}.$$

If all the coordinates are divided by this value, i.e., multiplied by 0.9397, the scale at 20° will equal E. That value is called k_0.

So if we set

$$\mathrm{E} = k_0 \mathrm{R}\lambda,$$
$$\mathrm{N} = k_0 \mathrm{R}\Phi,$$

and further multiply by the nominal scale of the map, this nominal scale will be the actual scale at $\phi = \pm 20°$, while the scale at the Equator will miss the nominal scale by a factor 0.9397.

Example: If the coordinates of a stereographic projection, according to the given formula, are multiplied by $k_0 = 0.94$, find the parallel for which the actual scale equals the nominal scale.

Taking $\mathrm{E} = 1$ (that is, forgetting about E), it is that parallel for which it would equal $1/k_0$ had we not multiplied by k_0:

$$k = \frac{2}{1 + \sin\phi} = \frac{1}{0.94},$$

hence

$$1 + \sin\phi = 1.88,$$
$$\phi = 61°38'32''5.$$

Therefore, the final value of k at a point of a map will equal

$$k_{\mathrm{Mercator}} = \frac{k_0}{\cos\phi}, \qquad k_{\mathrm{Stereog.}} = \frac{k_0}{1 + \sin\phi},$$

$$k_{\mathrm{Lambert}} = k_0 \frac{\cos\phi_0}{\cos\phi}\, e^{-\sin\phi_0(\Phi - \Phi_0)}.$$

The UTM projection

It is impossible to map a large area of the Earth without getting strong deformations. In order to solve this problem countries have chosen split projections; that is, a partition of their territory into several areas, each one being mapped with a different projection. An easy solution is to make use of several Lambert conic projections, each one with a different ϕ_0. These values can be separated by 10°, for instance. Even considering elliptic meridians, the formulas for the Lambert projection remain closed.

Another solution is the UTM projection. UTM stands for Universal Transverse Mercator. The Earth is now divided into sectors along meridians, each sector being 6° wide, and each of them is mapped by means of a Mercator projection, where the "equator" of the projection is the central meridian of the sector and the "poles" are the two points at the Earth Equator with λ equal to $\pm 90°$ of the central meridian's λ.

This gives raise to a very complicated projection, due to the fact that meridians are not circles but ellipses. Formulas are no longer closed expressions but instead Taylor series. Its advantage with respect to the zone division, as in the previous solution, is that formulas are the same in the different divisions. Nevertheless, many countries continued to use their old Lambert projections, which suited them better.

If the Earth were a sphere the transverse Mercator projection would amount to a change in coordinates $(\phi, \lambda) \to (\phi', \lambda')$, where the later are referred to a system where the equator is the central meridian of the sector and the "meridian" $\lambda = 0$ is the Earth Equator; plus a normal Mercator projection.

It is of interest to find an expression for the k of the projection as a function of the (E, N) coordinates. For spherical Earth this will be independent of whether the projection is the normal Mercator or the transverse one, and it is deduced as follows:

$$\mathrm{N} = k_0 \mathrm{R} \log \frac{1 + \sin\phi}{\cos\phi},$$

$$e^{\frac{\mathrm{N}}{k_0 \mathrm{R}}} = \frac{1 + \sin\phi}{\cos\phi}.$$

We have to find the value of $1/\cos\phi$ as a function of N. Let that value be written k', and let $\mathrm{N}/(k_0\mathrm{R})$ be written N':

$$e^{\mathrm{N}'} = k' \left(1 + \sqrt{1 - \cos^2\phi}\right) = k' + \sqrt{k'^2 - 1}.$$

Moving k' to the left hand side of the equation and squaring both sides we get

$$k'^2 - 2k' e^{\mathrm{N}'} + e^{2\mathrm{N}'} = k'^2 - 1,$$

$$2k' e^{\mathrm{N}'} = 1 + e^{2\mathrm{N}'},$$

$$k' = \frac{e^{-\mathrm{N}'} + e^{\mathrm{N}'}}{2} = \cosh \mathrm{N}'.$$

The value of k is $k_0 k'$, so we finally get

$$k = k_0 \cosh \mathrm{N}', \qquad \mathrm{N}' = \frac{\mathrm{N}}{k_0 \mathrm{R}}.$$

In the transverse projection the coordinates N and E are interchanged, so for the UTM with spherical Earth

$$k = k_0 \cosh \mathrm{E}', \qquad \mathrm{E}' = \frac{\mathrm{E}}{k_0 \mathrm{R}}.$$

The Earth mean radius of curvature along the direction perpendicular to the meridian is 6380 Km. The Taylor series for k taking this value of R and expressing E in thousands of kilometers is

$$k = k_0 \left\{ 1 + 0.01228 \left(\frac{\mathrm{E}}{k_0}\right)^2 + 2.5 \cdot 10^{-5} \left(\frac{\mathrm{E}}{k_0}\right)^4 \right\}$$

The (E, N) origin of the UTM coordinates is shifted so that there are no negative values. Points along the central meridian have $\mathrm{E} = 500\,000$ m, and a value of 10 000 000 is added to the N coordinates of the South hemisphere.

The actual formula for k taking the GRS80 ellipsoid, if E and N are expressed in meters, is

$$p = \frac{\mathrm{E} - 500\,000}{k_0\, 10^6}, \qquad q = \begin{cases} \dfrac{\mathrm{N}}{k_0\, 10^6}, & \text{North hemisphere} \\ \dfrac{\mathrm{N} - 10\,000\,000}{k_0\, 10^6}, & \text{South hemisphere} \end{cases}$$

$$k = k_0(1 + a_2 p^2 + a_4 p^4);$$

$$a_2 = 0.01237370 - 4.1275 \cdot 10^{-6} q^2 + 3.531 \cdot 10^{-8} q^4 - 1.347 \cdot 10^{-10} q^6 + 3.8 \cdot 10^{-13} q^8 - 10^{-15} q^{10},$$

$$a_4 = 2.621 \cdot 10^{-5} - 3.53 \cdot 10^{-8} q^2$$